Mathematical Appendix

Dan Elton

April 23, 2012

1 Art of the commutator

$$\begin{aligned}
[A, BC] &= [A, B]C + B[A, C] \\
[AB, C] &= A[B, C] + [A, C]B \\
[A, B^n] &= nB^{n-1}[A, B] \text{ if } [B, [A, B]] = 0
\end{aligned} \tag{1.1}$$

2 The Gamma Function

The Gamma function is a generalization of the factorial which appears often in physics (especially statistical mechanics). It is defined as :

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{2.1}$$

We perform integration by parts on this integral:

$$\begin{aligned}
\int_0^\infty t^{z-1} e^{-t} dt &= -t^{z-1} e^{-t}|_0^\infty - \int_0^\infty -(z-1) t^{z-2} e^{-t} dt \\
\Gamma(z) &= (z-1)\Gamma(z-1)
\end{aligned}$$

Leading to an important identity:

$$\Gamma(z+1) = z\Gamma(z) \tag{2.2}$$

Using induction on can prove the relation to the factorial:

$$\Gamma(n) = (n-1)! \tag{2.3}$$

A useful value of $\Gamma(z)$ is

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{2.4}$$

3 The Dirac δ Functional

The "Dirac delta function" is actually a functional (function that takes a function as an argument and returns a scalar), defined to have the following properties:

$$\begin{gathered}\int\limits_{-\infty}^{\infty} f(x)\delta(x-a) = f(a) \\ \int\limits_{-\infty}^{\infty} \delta(x)dx = 1\end{gathered} \tag{3.1}$$

3.1 Derivative of the δ functional

The derivative of the delta function can be easily proven by applying it to a test function $\phi(x)$ with compact support and using integration by parts:

$$\begin{aligned}\int\limits_{-\infty}^{\infty} \delta'(x)\phi(x)dx &= f(x)\phi(x)|_{-\infty}^{\infty} - \int\limits_{-\infty}^{\infty} \delta(x)\phi'(x)dx \\ &= -\int\limits_{-\infty}^{\infty} \delta(x)\phi'(x)dx \\ &= \phi'(0)\end{aligned} \tag{3.2}$$

3.2 Representations of the δ functional

Here you will find a collection of representations:

$$\begin{aligned}\delta(x) &= \lim_{k\to\infty} \frac{1}{2\pi} \int\limits_{-k}^{k} e^{ipx}dp \\ &= \lim_{\epsilon\to 0} \frac{1}{\pi} \frac{\epsilon}{x^2+\epsilon^2} \\ &= \lim_{k\to\infty} \frac{1}{\pi x} \sin\frac{kx}{2} \\ &= \lim_{\sigma\to 0} \frac{1}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{\sigma^2}} \\ &= \lim_{t\to\infty} \frac{\sin^2 xt}{\pi t x^2} \\ &= \lim_{\epsilon\to 0} \frac{1}{\pi x} \sin\left(\frac{x}{\epsilon}\right) \\ &= \lim_{\epsilon\to 0} \frac{1}{2}\epsilon|x|^{\epsilon-1}\end{aligned} \tag{3.3}$$

3.3 Other δ-functional properties

In QFT one often encounters δ functionals with a function as an argument. First consider this identity:

$$\delta(ax) = \frac{1}{|a|}\delta(x) \tag{3.4}$$

This can be proven by considering changing the integration variables so $y \equiv ax, dx\ dx = \frac{1}{a}dy$. More generally we can consider $\delta(f(x))$.

$$\delta(f(x)) = \sum_i \frac{1}{f'(a_i)}\delta(x - a_i) \tag{3.5}$$

where a_i are the zeros of the function f. This formula can be proven by Taylor expanding $f(x)$ around each of it's zeros. Note that this formula diverges if the zeros are non-simple, ie. if $f'(a_i) = 0$.

Using integration by parts we can arrive at this equation:

$$\int f(x)\delta^n(x)dx = -\int \frac{f}{x}\delta^{n-1}dx \tag{3.6}$$

and the related relation:

$$x^n\delta^n = (-1)^n n!\delta(x) \tag{3.7}$$

3.4 Fourier expansion of the δ-functional

This is a curious formula I found in a math appendix by Marina von Steinkirch and also on Wolfram Mathworld:

$$\begin{aligned} \delta(x - a) &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}[\cos(na)\cos(nx) + \sin(na)\sin(nx) \\ &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos[n(x-a)] \end{aligned} \tag{3.8}$$

It doesn't seem like it would work, but somehow it does..

4 Gaussian Integrals

Here you will find Gaussian integrals, including integrals that regularily appear in field theory. The fundamental gaussian integral is:

$$\int\limits_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}} \tag{4.1}$$

The formula for the integration of a gaussian times a power of x can be derived by taking the fundamental formula and differntiating both sides with respect to a n times.

$$\int_{-\infty}^{\infty} x^n e^{-ax^2} dx = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \sqrt{\frac{2\pi}{a^n}}(2n-1)!! & \text{if } n \text{ is even} \end{cases} \tag{4.2}$$

The "double factorial" !! is defined as $n!! = n(n-2)(n-4)\cdots 5 \cdot 3 \cdot 1$.

Next we present the equation for a Guassian with a "linear source function", J, which can be proven by completing the square of the exponent. In the following equations we include a factor of $1/2$ with x^2 since that is what usually appears in field theory.

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2+Jx} dx = \sqrt{\frac{2\pi}{a}} e^{J^2/2a} \tag{4.3}$$

When analyzing path integrals, we need to generalize this to repeated integration.

5 Fourier transform stuff

$$\hat{C}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \exp(-i\omega t) dt \tag{5.1}$$

5.1 Orthogonality relations

$$\int_{-\infty}^{+\infty} \exp(-i\omega t) \exp(-i\omega t') d\omega = 2\pi\delta(t-t') \tag{5.2}$$

5.2 Convolution/correlation theorem

If

$$\hat{f}(t) = \int_{-\infty}^{+\infty} A(t')B(t-t')dt' \tag{5.3}$$

Then

$$\hat{C}(\omega) = \hat{A}(\omega)\hat{B}(\omega) \tag{5.4}$$

similarily, if

$$C(t) = \int_{-\infty}^{+\infty} A(t')B(t+t')dt' \tag{5.5}$$

Then

$$\hat{C}(\omega) = \hat{A}(-\omega)\hat{B}(\omega) = \hat{A}^*(\omega)\hat{B}(\omega) \tag{5.6}$$