Maxwell's Equations

Dan Elton

April 23, 2011

Mawell's orginal equations ( )

Maxwell's Equations in Differential Form (SI units)

$$\boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{E}=\dfrac{\rho}{\epsilon_0} & \vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{B}=0 & \vec{\nabla}\times\vec{B}=\mu_0 J+\dfrac{1}{c^2}\dfrac{\partial\vec{E}}{\partial t} \end{array}}$$

Maxwell's Equations in Integral Form (SI units)

$$\boxed{\begin{array}{ll} \displaystyle\int_{\partial V}\vec{E}\cdot d\vec{A}=\int_V\frac{\rho}{\epsilon_0}dV & \displaystyle\int_{\partial A}\vec{E}\cdot d\vec{l}=-\frac{\partial}{\partial t}\int_A\vec{B}\cdot d\vec{A} \\ \displaystyle\int_{\partial V}\vec{B}\cdot d\vec{A}=0 & \displaystyle\int_{\partial A}\vec{B}\cdot d\vec{l}=\mu_0\int_A\vec{j}\cdot d\vec{A}+\frac{1}{c^2}\frac{\partial}{\partial t}\int_A\vec{E}\cdot d\vec{A} \end{array}}$$

Potential formulation

$$\vec{E}\equiv-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}$$
$$\vec{B}\equiv\vec{\nabla}\times\vec{A}$$

$$\boxed{\begin{array}{c} \nabla^2\phi+\dfrac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}=-\dfrac{\rho}{\epsilon_0} \\ \nabla^2\vec{A}-\dfrac{1}{c^2}\dfrac{\partial^2\vec{A}}{\partial t^2}-\vec{\nabla}(\vec{\nabla}\cdot\vec{A}+\dfrac{1}{c^2}\dfrac{\partial\phi}{\partial t})=-\mu_0\vec{J} \end{array}}$$

Four-potential formulation

$$J^\mu \equiv (\rho c, \vec{j})$$
$$A^\mu \equiv (\phi, c\vec{A})$$
$$\Box = \partial_\mu \partial^\mu \equiv \frac{\partial^2}{\partial t^2} - \nabla^2$$
$$\partial_\mu A^\mu = 0$$

$$\boxed{\Box A^\mu = \mu_0 J^\mu}$$

Covariant Notation

$$F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha$$
$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_\mathrm{z} & -B_\mathrm{y} \\ E_y/c & -B_\mathrm{z} & 0 & B_\mathrm{x} \\ E_z/c & B_\mathrm{y} & -B_\mathrm{x} & 0 \end{pmatrix}$$

$$\boxed{\begin{gathered} \partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta \\ \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \end{gathered}}$$

Covariant Notation with the Dual Tensor

$$G^{\alpha\beta} \equiv \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

$$\boxed{\begin{gathered} \partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta \\ \partial_\alpha G^{\alpha\beta} = 0 \end{gathered}}$$

Lagrangian Mechanics Formulation (SI Units)

$$\boxed{\begin{gathered}\mathcal{L}_{EM} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \\ \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} - \frac{\partial \mathcal{L}}{\partial A^\nu} = 0\end{gathered}}$$

Differential Geometry formulation

2x2 Matrix Notation

$$\mathbf{F} \equiv \mathbf{E} + i\mathbf{B} = \begin{pmatrix} E_3 & E_1 - iE_2 \\ E_1 + iE_2 & -E_3 \end{pmatrix} + i \begin{pmatrix} B_3 & B_1 - iB_2 \\ B_1 + iB_2 & -B_3 \end{pmatrix}$$

$$j \equiv \rho + \vec{j}$$

$$\boxed{\bar{\partial} F = \frac{1}{\epsilon_0} \bar{j}}$$

The Constituiative Relations

Maxwell's equations can be recast in the common "macroscopic form", where $\vec{B}$ is replaced by what is normally called the "magnetic field", (or "magnetic displacement") , $\vec{H}$ and the $\vec{E}$ is replaced by the Electric Displacement $\vec{D}$.

$$\begin{aligned}\vec{E} &= \frac{1}{\epsilon_0}(\vec{D} - \vec{P}) \\ \vec{H} &= \mu_0(\vec{H} + \vec{M})\end{aligned} \tag{0.1}$$

Converting from SI to Gaussian

As explained in Appendix A of the 2nd edition of [**?**], the Gaussian or cgs system of units is designed so that the Columb constant $k = 1$. However, it is very difficult to measure the force between charged spheres in the laboratory to high accuracy. Thus the traditional SI units system is based of curent measurement. The force formula which is "fundamental" to the SI system is :

$$\frac{F_I}{L} = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d} \tag{0.2}$$

The ampere is defined as "that steady current which, when present in each of two long parallel conductors, seperated by a distance d of one meter, results in a force per meter of length between them numerically equal to 2×10^{-7} N/m." Thus under this definition, μ_0 is precisely defined as:

$$\mu_0 \equiv 4\pi \times 10^-7 \tag{0.3}$$

When converting Gaussian to SI, one makes the following conversions:

$$\begin{aligned} \rho &\to \frac{1}{\sqrt{4\pi\epsilon_0}}\rho \\ \vec{E} &\to \sqrt{4\pi\epsilon_0}\vec{E} \\ \vec{B} &\to c\sqrt{4\pi\epsilon_0}\vec{B} \end{aligned} \tag{0.4}$$

There is also the "Rationalized Heaveside-Lorentz" units system, which is the same as cgs, but involves a rescaling of $\vec{E}$ and $\vec{B}$ to remove the factors of 4π and sets $c = 1$.