Classical Mechanics (2014)

\[ W = \int_{x_1}^{x_2} F(x) \, dx \]

\[ \mathbf{F} = -\nabla V \]

**Impulse**

\[ \mathbf{J} = \int_{t_1}^{t_2} \mathbf{F} \, dt = \mathbf{F} \Delta t = \Delta \mathbf{P} \]

**Center of mass**

\[ \mathbf{C}_M = \frac{1}{M} \sum_{i=1}^{n} m_i \mathbf{r}_i \]

**Air resistance**

- Low velocity: \( \mathbf{F}_r = -b \mathbf{v} \)
- High velocity: \( \mathbf{F}_r = -c \mathbf{v}^2 \)

**Rocket motion**

\[ F_{\text{thrust}} = \frac{dp}{dt} = V_{\text{exhaust}} \frac{dm}{dt} \]
Collisions

Elastic collisions

\[ V_{1f} = \left( \frac{m_1 - m_2}{m_1 + m_2} \right) V_{1i} + \left( \frac{2m_2}{m_1 + m_2} \right) V_{2i} \]

\[ V_{2f} = \left( \frac{2m_1}{m_1 + m_2} \right) V_{1i} + \left( \frac{m_2 - m_1}{m_1 + m_2} \right) V_{2i} \]

Notes:
- In the case \( m_1 = m_2 \),
  \[ V_{1f} = V_{2i} \]
- In the case \( m_2 \gg m_1 \), \( V_{2i} \approx 0 \),
  \[ V_{2f} \approx -V_{1i} \]

In elastic collisions, the relative velocities of the particles is equal and opposite to the relative velocities afterwards. (\( \alpha = V_{1f} - V_{2i} \) is conserved)
Gravitation
\[ F = \frac{G m_1 m_2}{r^2} \]

Kepler's 1st Law
- Perihelion or perigee
- Aphelion (if sun at focus)
- Apogee (if earth at focus)

Kepler's 2nd Law
\[ \frac{dA}{dt} \to 0 = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{2} r \cdot \dot{r} \cdot \Delta \theta = \frac{1}{2} r^2 \omega \]

\[ L = mvr = mr^2\omega = \text{const.} \]

Kepler's 3rd Law
\[ T^2 = \left( \frac{4\pi^2}{GM} \right) r^3 \]

\[ T^2 \propto r^3 \]

Eccentricity
- \( e = 0 \) circle
- \( 0 < e < 1 \) ellipse
- \( e = 1 \) parabola
- \( e > 1 \) hyperbola
Sound Waves

\[ V_{sound \, air} = 343 \text{ m/s} = \text{1 mile / 5 seconds} \]

\[ V_{sound \, air} \propto \sqrt{T} \]

Closed pipe

\[ \lambda = \frac{2L}{n} \quad n = 1, 2, 3, \ldots \]

Open pipe

\[ \lambda = \frac{4L}{n} \quad n = 1, 2, 3, \ldots \]

Doppler effect

\[ V'_0 = \text{source frequency} \]
\[ V' = \text{observer frequency} \]
\[ V = \text{speed of sound} \]
\[ V_s = \text{speed of source} \]
\[ V_o = \text{speed of observer} \]

master

\[ V' = V_0 \frac{V \pm V_o}{V \mp V_s} \]

observer at rest

\[ V' = V_0 \frac{V}{V \mp V_s} \]
Angular Momentum

\[ L = m \vec{\omega} \times \vec{r} = \vec{p} \times \vec{r} = \vec{r} \times \vec{p} = m r \vec{v}_r = I \omega \]

Rotational Analogs

\[ K_{rot} = \frac{1}{2} I \omega^2 \]

\[ \tau = I \alpha \]

Moments of Inertia

- \[ I = m R^2 \] hoop
- \[ I = \frac{m R^2}{2} \] disk/cylinder
- \[ I = \frac{m (R_1^2 + R_2^2)}{2} \] hollow disk/cylinder
- \[ I = \frac{m L^2}{12} \] rod through center of mass
- \[ I = \frac{m L^2}{3} \] rod through one end
- \[ I = \frac{m (a^2 + b^2)}{12} \] solid plate
- \[ I = \frac{2}{5} m R^2 \] sphere

*Parallel Axis Theorem*

\[ I = I_{cm} + md^2 \]
Classical Mechanics: Oscillations

**Simple Harmonic Oscillator (SHO)**

\[ \ddot{x} + \frac{k}{m} x = 0 \]

\[ \omega^2 = \frac{k}{m} \]

\[ f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \]

**Simple Pendulum**

\[ \tau = I \dot{\theta} \sin \theta = m x^2 \dot{\theta} = I \dot{\theta} \]

\[ \Rightarrow \ddot{\theta} + \frac{g}{x} \sin \theta = 0 \quad \text{or} \quad \ddot{\theta} + \frac{k_m g \sin \theta}{I} = 0 \]

\[ T = 2\pi \sqrt{\frac{I}{mgL}} \]

**Physical Pendulum**

\[ T = 2\pi \sqrt{\frac{I}{mgL}} \]

**Forced/Damped Oscillations**

\[ m \ddot{x} + b \dot{x} + kx = 0 \]

\[ \ddot{x} + 2\beta \dot{x} + \omega_0^2 x = f(t) \cos \omega t \]

\[ \Rightarrow x(t) = A \cos (\omega t - \delta) \]

\[ A = \sqrt{\frac{f_c^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2}} \]

\[ \delta = \arctan \left( \frac{2\beta \omega}{\omega_0^2 - \omega^2} \right) \]
Fluid statics

\[ P = \frac{dF}{dA} \]

Pascal's principle: basis for hydraulics
Archimedes principle: buoyant force = weight of displaced water

Manometer

\[ P_0 \]

\[ P - P_0 = \rho g h \]

Fluid Dynamics

Continuity Equation

\[ \rho \nabla \cdot \nabla + \frac{\partial P}{\partial t} = 0 \]

Bernoulli's Equation

\[ P + \frac{1}{2} \rho v^2 + \rho gh = \text{constant} \]
Intermediate Mechanics

\[ L = T - U \]
\[ H = T + U \]

\[ L = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = H - \sum_j p_j \dot{q}_j \]

Lagrange-Euler Equation (w/ Lagrange multipliers)

\[ \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \sum_k \lambda_k \frac{\partial \mathcal{F}}{\partial q_k} = 0 \]

Hamilton's Equations

\[ \dot{q}_j = \frac{\partial H}{\partial p_j} \]
\[ \dot{p}_j = -\frac{\partial H}{\partial q_j} \]
The Electric Field

\[ E = \frac{\vec{E}}{q} \]

\[ \vec{D} = \varepsilon_0 \vec{E} = \vec{P} \]

Electric Field of a Dipole

\[ E = \frac{1}{4\pi \varepsilon_0} \frac{q_{d}}{\left(x^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}} \approx \frac{1}{4\pi \varepsilon_0} \frac{q_{d}}{x^3} \text{ when } x \gg d \]

Dipole moment

\[ \vec{p} = q \vec{d} \]

Electric dipole torque

\[ \vec{\tau} = \vec{p} \times \vec{E} \]

Electric Field of an Infinite sheet

\[ E_x = \frac{\sigma}{2\varepsilon_0} \]

Electric Potential Energy

\[ U(n) = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{r} \]

Electric Potential

\[ V = \frac{U}{q_0} \]

\[ \Delta V = \int_{a}^{b} \vec{E} \cdot d\vec{s} \iff \vec{E} = -\nabla V \]
Capacitance

\[ C = \frac{Q}{V} \]

Parallel Plate Capacitor

\[ C = \frac{\varepsilon_0 \ A}{D} \quad V = \int_{A} E \cdot ds = \frac{Q}{\varepsilon_0} \]

\[ E = \frac{\sigma}{2\varepsilon} + \frac{\varphi}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0} \cdot \frac{A}{\varepsilon_0} \]

Capacitors in Parallel

\[ C = C_1 + C_2 + \ldots \]

Capacitors in Series

\[ \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots \]

Energy Stored in a Capacitor

\[ U = \frac{1}{2} CV^2 \quad = \frac{1}{2} \frac{Q^2}{C} \]

Capacitor with dielectric

\[ C = \varepsilon K C_0 \]

Ohm's Law

\[ V = IR \quad I = \frac{V}{R} \quad R = \frac{V}{I} \]

Resistance

\[ \rho = \frac{E}{J} = \frac{E}{I/A} \]

\[ \rho = \frac{RA}{L} \]

\[ R = \rho \frac{L}{A} \]
Inductance

\[ \mathcal{E} = L \frac{di}{dt} \quad \xrightarrow{\text{Faraday's Law}} \quad L = \frac{\Phi}{i} \]

Energy of an inductor

\[ U = \frac{1}{2} Li^2 \]

Electric Filters

- Low pass filters
  \[ \text{Vin} \rightarrow \text{Vout} \]

- High pass filters
  \[ \text{Vout} \]

Current Density

\[ j = \frac{i}{A} = V_d \frac{n}{e} \]

Hall effect coefficient

\[ R_H = -\frac{1}{he} = \frac{E_d}{jX} \]

Impedance / AC Circuit Analysis

\[ Z_{\text{inductor}} = j\omega L \quad Z_{\text{capacitor}} = -\frac{1}{j\omega C} \quad Z_{\text{resistor}} = R \]

Impedance:

\[ V = jZ \]

defined as \( V / i \), as a function of \( \omega \). It is analogous to resistance with DC.

Reactance:

Is the complex part of impedance and results in a phase shift of the AC signal.

Resonant:

 '\text{see next page}'

Master equation:

\[ Z = R + j\omega L - j\omega C \]
Electric Potentials

\[ \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \]

\[ \vec{B} = \nabla \times \vec{A} \]

Energy of E & B Fields

\[ U = \frac{1}{2} \int_V (\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2) dV \]

Momentum of E & B Fields

\[ \vec{p} = \varepsilon_0 \int_V \vec{E} \times \vec{B} dV \]

Poynting Vector

\[ \vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) \]

Biot - Savart Law

\[ \vec{B} = \frac{\mu_0}{4\pi} \int \frac{I \, d\vec{l} \times \vec{r}}{r^2} = \frac{\mu_0}{4\pi} \int \frac{I \, d\vec{l} \times \vec{r}}{r^3} \]

Larmor Formula

\[ \vec{v} = \omega_0 \times \vec{A} \]

\[ P \propto q^2 a^2 \]
Maxwell's Equations

Gauss's Law
\[ \oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enc.}}}{\varepsilon_0} \]
\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]

No Magnetic Monopoles
\[ \oint \mathbf{B} \cdot d\mathbf{A} = 0 \]
\[ \nabla \cdot \mathbf{B} = 0 \]

Faraday's Law
\[ \oint \mathbf{E} \cdot d\mathbf{L} = -\frac{\partial \Phi}{\partial t} \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

Ampere's Law
\[ \oint \mathbf{B} \cdot d\mathbf{L} = \mu_0 i_{\text{enc.}} + \mu_0 \varepsilon_0 \frac{\partial \Phi}{\partial t} \]
\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \]

AC Circuit Analysis

LRC Circuit

Resonance occurs when the reactance (complex impedance) is 0.
\[ Z = R + i(\omega L - 1/j\omega C) \Rightarrow \omega L = 1/j\omega C \Rightarrow C = \frac{1}{\omega^2} \]

\[ \omega = \sqrt{\frac{1}{LC} - \frac{1}{\omega C}} \]

Mechanical Analog
\[ m\ddot{x} + b\dot{x} + kx = 0 \]
\[ Lq'' + Rq' + \frac{q}{C} = 0 \]
\[ V = I \cdot R \]

Power = \( I^2 \cdot R \)

\[
\frac{dQ}{dt} + qC = V \\
Q = CV \left[ 1 - e^{-\frac{t}{RC}} \right]
\]

\[
\frac{dQ}{dt} = V \\
\frac{dQ}{dt} + \frac{Q}{L} = V
\]

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**Lenz's Law**

"An induced current is always such to oppose the changes causing it."

**Right-Hand Rules**

- Induced current direction
- Increasing \( B \)
- Decreasing \( B \)

- Induced current direction
- Increasing \( B \)
- Decreasing \( B \)
Electric boundary conditions for reflection from an infinitely long, perfectly conducting sheet:

\[ E_\parallel = 0, \quad B_\perp = 0 \]

For a perfect conductor, there can never be an E-field parallel to the surface. The phase of an incoming wave shifts by π, so the E-field reverses direction, canceling out.

Cyclotron frequency:

\[ \omega = \frac{qB}{m} \]
Thin Lens Equation

\[ \frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i} \]

\( d_o = \text{object distance} \)
\( d_i = \text{image distance} \)

Rayleigh Criterion

\[ \theta = \frac{1.22\lambda}{D} \]

Malthus's law for Polarizers

\[ I(\theta) = I(0) \cos^2 \theta \]

Light Intensity

\[ I = \varepsilon_0 c \langle E^2 \rangle_t \]

Snell's Law

\[ n_1 \sin \theta_1 = n_2 \sin \theta_2 \]

Total internal reflection

\[ \theta = \arcsin \left( \frac{n_2}{n_1} \right) \]
Brewster's angle
\[ \theta_B = \arctan \left( \frac{\theta_0}{\theta_1} \right) \]

Optical Path Length
\[ opl = \int n \, dl = \sum n_i \, dl \]

Maxwell's Relation
\[ n \approx \sqrt{k} \]

Double Slit interference /
\[ ds \sin \theta = m \lambda \quad m = 0, 1, 2, \ldots \quad (\text{maxima}) \]
\[ \frac{y}{y_0} = \frac{m \lambda D}{D} \quad \text{for maxima} \]
Phase change after reflectance \( \approx \pi \) (from lower to higher \( n \))

Bragg Diffraction
Waves are in phase when
\[ 2ds \sin \theta = n \lambda \quad n = 0, 1, 2, \ldots \]

Other diffraction problems
1-slit diffraction minima
\[ W \sin \theta = m \lambda \]
Circular diffraction, first minima
\[ \sin \theta \approx 1.22 \frac{\lambda}{D} \quad (\text{same as Rayleigh criterion}) \]

Diffraction grating as well
\[ \text{double-slit diffraction} \]
\[ \text{minima:} \quad m \sin \theta = m \lambda \]
"missing fringes" occurs when diffraction minima cancel interference maxima in the double slit experiment

Thin films
\[ n_1, n_2 \]
General principle: phase change occurs when \( n_2 > n_1 \), does not occur when \( n_2 < n_1 \),
Constructive interference \( 2t = \frac{\lambda}{2} \)
Destructive interference \( 2t = \lambda \)
Thermodynamics: 10%

Heat Capacity

\[ C = \frac{\Delta Q}{\Delta T} \]

Specific Heat Capacity

\[ c = \frac{1}{m} \frac{\Delta Q}{\Delta T} \]

Mayer's Equation

\[ C_p - C_v = R = \frac{\hbar}{Nk} \]

Ideal Gas Law

\[ PV = nRT \quad \text{or} \quad PV = NkT \]

Work

\[ W = -\int P \, dV \quad \text{general equation} \]

constant volume: \( W = 0 \)

constant pressure: \( W = -P (V_f - V_i) \)

adiabatic: \( PV^\gamma = \text{const.} \rightarrow W = \frac{1}{\gamma-1} \left( P_f V_f - P_i V_i \right) \)

constant temperature: \( W = -nRT \ln \left( \frac{V_f}{V_i} \right) \)

Thermal Expansion

\[ \Delta L = \alpha \cdot L \cdot \Delta T \]

Speed of sound

\[ V_s \propto T^{\frac{1}{2}} \]
Entropy
\[ ds = \frac{dQ}{T} \]

Boltzmann's Law
\[ S = k \log w \]
\[ w = \text{# of possibilities} \]

Law of Dulong & Petit
\[ C_v = 3R = 3Nk \]
\[ \text{at high } T \]

Debye \( T^3 \) Law
\[ C_v \propto T^3 \]
\[ \text{at lower } T \]

*Equipartition Theorem*
\[ E = \frac{1}{2} n k T \]

Specific heat of a metal
\[ c = a T + B T^3 \]
\[ c = e^{aT} \] (superconducting)

Stirling's Theorem/approximation
\[ \ln N! \approx N \ln N - N \quad N \gg 0 \]
Internal energy of a gas (partition theorem)

\[ U = \left( \frac{1}{2} n k T \right) N \]

- \( n \) = degrees of freedom
- \( N \) = number of atoms/molecules

Van der Waals equation

\[(p + a \frac{n^2}{V^2})(V - nb) = nRT\]

Maxwellian speed distribution

\[ n(v) = 4\pi N \left( \frac{m}{2\pi kT} \right)^{3/2} v^2 e^{-mv^2/2kT} \]

Maxwell-Boltzmann energy distribution

\[ n(E) = \frac{2n}{\pi^{3/2}} \frac{1}{(kT)^{3/2}} E^{1/2} e^{-E/2kT} \]

Bose-Einstein distribution

\[ f_{BE}(E) = \frac{1}{e^{(E-E_0)/kT} - 1} \]

- indistinguishable
- no Pauli Exclusion

Fermi-Dirac distribution

\[ f_{FD}(E) = \frac{1}{e^{(E-E_0)/kT} + 1} \]

- indistinguishable
- Pauli Exclusion principle

Boltzmann Distribution / Maxwell-Boltzmann

\[ f_j = \frac{N e^{-\varepsilon_j / kT}}{\sum_j g_j e^{-\varepsilon_j / kT}} \]

- distinguishable
Blackbody Planck Formula

\[
\frac{\text{Power}}{\text{Area} \cdot \lambda} = \frac{\text{Flux}}{d\lambda} = \frac{dR}{d\lambda} = \frac{2\pi hc^2}{\lambda^5(e^{hc/\lambda kT} - 1)}
\]

Stephan-Boltzmann Law

\[ R = \text{Power Output} = \sigma T^4 \]

Wein's Law for Blackbody

\[ \lambda_{\text{max}} \propto \frac{1}{T} \]

First Law of Thermodynamics

A system goes from state \( i \) to \( f \) through various paths, the quantity \( \Delta Q + \Delta W \) is always the same.

\[ \Delta E = \Delta Q + \Delta W \]

Second Law of Thermodynamics

There are no perfect heat engines.

\[ \rightarrow \text{It is not possible for a cyclical process to convert heat entirely to work.} \]
\[ \rightarrow \text{It is impossible to build a heat engine more efficient than a Carnot engine, } \]
\[ e_{\text{Carnot}} = 1 - \frac{T_c}{T_H} = \frac{T_H - T_c}{T_H} \]

\[ \rightarrow \text{A perfectly reversible engine has Carnot efficiency.} \]
\[ \rightarrow \text{It is impossible to reach absolute zero. (Sometimes called "3rd law")} \]

Zeroth Law of Thermodynamics

If \( A \) & \( B \) are in \( T_i, E_j \) with \( C_j \),
then \( A \) and \( B \) are in \( T_i, E_j \) with each other.
Fourier's Law of Heat Conduction

\[ \dot{q} = -k \nabla T \]

where:
- \( \dot{q} \) = heat flux, \( \dot{q} = \frac{dq}{dt} \)
- \( k \) = thermal conductivity
- \( T \) = temperature

\[ \frac{d\dot{q}}{dt} = -kA \frac{dT}{dx} \]

Von 1st Law of Thermodynamics / Energy

\[ dE = dQ + dW \]

\[ dU = TdS - PdV + \mu d\mu \quad (\mu = \text{chemical potential}) \]

The Partition Function

\[ Z = \sum_j g_j e^{-\epsilon_j/kT} \]

\[ S = \frac{U}{T} + Nk(\ln Z - \ln N + 1) \]

\[ F = -NkT(\ln Z - \ln N + 1) \]

\[ U = NkT^2 \left( \frac{\partial \ln Z}{\partial T} \right) \]

\[ \dot{H} = U + PV \quad \dot{H} = TdS - PdV \]

\[ F = U - TS \quad \dot{H} = \dot{S} + VdP \]

RMS Speed: Quick derivation

\[ \frac{3}{2} kT = \frac{1}{2} m \nu^2 \]

\[ \nu_{\text{rms}} \approx \sqrt{\frac{3kT}{m}} \]
Graham's Law of Effusion

\[
\frac{\text{Rate}_1}{\text{Rate}_2} = \sqrt{\frac{M_2}{M_1}} \quad \text{molar mass}
\]

can be derived by simply noting that \( V \propto \sqrt{\frac{1}{m}} \).
Quantum Physics (12.4)

The Schrödinger Equation

\[ -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi = i\hbar \frac{\partial \psi}{\partial t} \]

Operators

\[
\begin{align*}
\hat{x} &= x \\
\hat{p} &= -i\hbar \frac{\partial}{\partial x} \\
\hat{E} &= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}
\end{align*}
\]

De Broglie Formulae

\[ E = \frac{p^2}{2m} \]

Planck Formulae (only applies to photons!)

\[ E = h\omega = hf = \frac{hc}{\lambda} \]

Heisenberg's Uncertainty Relations

\[ \sigma_x \sigma_p \geq \frac{\hbar}{2} \]

\[ \sigma_t \sigma_E \geq \frac{\hbar}{2} \]

Ehrenfest's Theorem

\[ \frac{\partial \langle \hat{p} \rangle}{\partial t} = -\frac{\partial \langle \hat{V} \rangle}{\partial x} \]

Compton Scattering Formula

\[ \Delta \lambda = \frac{hc}{mc^2} (1 - \cos \theta) \]

\[ \lambda_{\text{Compton}} = \frac{hc}{mc^2} = \frac{h}{mc} \]

TISE

\[ -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi = E \psi \]

Time Dependence

\[ f(t) = e^{-iEt/\hbar} \]

\[ t \rightarrow \text{missing} \]
Quantum Physics Continued

Infinite Square Well

\[ E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{2ma^2} \]

\[ \Psi = \sqrt{\frac{2}{a}} \sin \left( \frac{n \pi}{a} x \right) e^{-iEt/\hbar} \]

The Harmonic Oscillator

\[ -\frac{\hbar^2}{2m} \nabla^2 \Psi + \nu \Psi = E \Psi \]

\[ -\frac{\hbar^2 \omega^2}{2m} \Psi + \frac{1}{2} m \omega^2 x^2 \Psi = E \Psi \quad \omega = \frac{k}{m} \]

\[ \frac{1}{2m} \left[ \left( \frac{\hbar^2}{i \hbar} \frac{d}{dx} \right)^2 + (m \omega x)^2 \right] \Psi = E \Psi \]

\[ a_+ \equiv \frac{1}{i \hbar \nu} \left( \frac{\hbar^2}{i \hbar} \frac{d}{dx} + i m \omega x \right) \]

\[ (a_+ a_- - \frac{1}{2} \hbar \nu) \Psi = E \Psi \]

\[ \Psi_0 = A_0 e^{-\frac{m \omega}{2 \hbar} x^2} \]

\[ E_n = (n + \frac{1}{2}) \hbar \omega \]

The Free Particle

Ψ(x,t) = Ae^{ik(x - \frac{\hbar k}{2m} t)} + Be^{-ik(x + \frac{\hbar k}{2m} t)}

Fourier Transform

\[ \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x_0) e^{-ikx} dx \]
**Group Velocity**

\[ V_{\text{group}} = \frac{\omega}{\partial k} \]

**Phase Velocity**

\[ V_{\text{phase}} = \frac{\omega}{k} \]

**Delta Function**

\[ \int_{-\infty}^{\infty} f(x) \delta(x-a) \, dx = f(a) \]

**Basic Definitions**

Hermitian operator: \( T^+ = T \)

Unitary operator: \( U^+ = U^{-1} \)

**Probability Current**

\[ \frac{\partial \rho_{ab}}{\partial t} = J(a,t) - J(b,t) \]

\[ J(x,t) = \frac{\hbar}{2m} \left( \frac{\partial \Psi^* \Psi}{\partial t} - \frac{\partial \Psi^* \Psi}{\partial x} \right) \]

**Commutator Relations**

\[ [A, B] = [-B, A] \]

\[ [A, B] = AB - BA \]


\[ AC - AB = [A, B] C \]

\[ AB - AC = [A, C] B \]
Spin & Angular Momentum

\[ L^2 \psi = \hbar^2 L(l+1) \psi \]
\[ L_z \psi = m \hbar \psi \]

Pauli matrices

\[ \sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
\[ \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]
\[ \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

Hydrogen atom

\[ E = -13.6 \left( \frac{L^2}{n^2} - \frac{1}{n^2} \right) \]

To derive, assume that \( m e v r = n \hbar \)

Bohr radius \( a_0 = \frac{4 \pi e^2 \hbar^2}{m e^2} \)

Angular Momentum quantum numbers

- \( n \): principal quantum number
- \( \ell \): Azimuthal quantum number - gives orbital angular momentum
- \( m_e \): magnetic quantum number, ranges from \(-\ell\) to \(\ell\)
- \( s \): spin quantum number

For electrons: \[ \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2} \]

\[ m_s = m \hbar \]

\[ m_s = m \hbar \text{ for fermions, } \]

\[ -\frac{1}{2} \text{ to } \frac{1}{2} \text{ in } \mathbb{Z} \]
Positional
\[ \mu = \frac{me^2}{me + me} = \frac{e^2}{2m} = \frac{e^2}{2} \Rightarrow E \text{ levels are } \frac{1}{2} \text{ of hydrogen's.} \]

The Planck Length
\[ L_p = 7 \frac{Gh}{c^3} \]

Schwarzschild Radius
\[ R_s = \frac{2MG}{c^2} \]

Constants
\[ kT \approx \frac{1}{40} \text{ eV at } 300\text{K} \]
\[ h\nu = 1.240 \text{ eV nm} \]
\[ G = 6.67 \times 10^{-11} \]
\[ K = 8.62 \times 10^{-5} \text{ eV K} \]

Photoelectric Effect
\[ E = eV_{\text{stop}} = hf - \phi \]

Bohr Atom
\[ \Delta E = -2 \pi Z^2 \frac{\hbar^2}{13.6} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \]

Spectroscopic Notation / "Term Symbol"
\[ L = \ell = \frac{p_{\phi}}{\hbar} \]
\[ S = s_{\phi} \]
\[ J = \ell + S \]
\[ L = \ell + \ell + \cdots \]
\[ J = S + L \]

for example
\[ 2S = 3 \]
\[ 2 \ell + S = 3 \]
\[ \ell = 0, 1 \]
\[ S = 0, 1 \]
\[ J = 1, 2 \]
MISC.

Intrinsic Magnetic Moment

\[ \vec{\mu}_s = \gamma \vec{s} \]

\[ \gamma = \frac{e g}{2 m} \]

A useful rule of thumb for Energy levels

Electronic transitions \( \approx 1 \text{ eV} \)

Vibrational transitions \( \approx 0.1 \text{ eV} \)

Rotational transitions \( \approx 0.001 \text{ eV} \)
**Special Relativity** (6%)  
\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad \beta = \frac{v}{c} \]

\[ \gamma (\beta = 1/3) = 1.15 \]
\[ \gamma (\beta = 0.9) = 2.3 \]

**Energy-momentum**

\[ E^2 = p^2 c^2 + m^2 c^4 \quad \text{*master equation*} \]

\[ E = \gamma mc^2 \quad \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \]
\[ p = \gamma mv \]

\[ KE = \gamma mc^2 - mc^2 \]

For a photon, \( m = 0 \) and
\[ E = pc \]

**Lorentz transformation**

\[ x' = \gamma (x - vt) \]
\[ y' = y \]
\[ z' = z \]
\[ t' = \gamma (t - \frac{xv}{c^2}) \]
Transformation of velocities

\[ u' = \frac{dx'}{dt'} = \text{velocity in frame } S' \]
\[ u' = \frac{dx'}{dt'} = \text{velocity in frame } S' \]
\[ S' \text{ is moving with velocity } v \]

\[ u' = \frac{u - v}{1 - \frac{uv}{c^2}} \]

The invariant / spacetime interval / proper distance

\[ ds^2 = dr^2 - c^2 dt^2 \]

Time dilatation

\[ t = \gamma t_0 \]

Length contraction

\[ \ell = \frac{\ell_0}{\gamma} \]

The Doppler Shift

\[ E' = \gamma E_0 + \beta E \]

Blue shift:
\[ \frac{\lambda'}{\lambda} = \sqrt{\frac{1 - \beta}{1 + \beta}} \]

Red shift:
\[ \frac{\lambda'}{\lambda} = \sqrt{\frac{1 + \beta}{1 - \beta}} \]
Standard deviation

$$\sigma^2 = (X - \bar{X})^2 = \langle (X - \langle X \rangle)^2 \rangle = \langle X^2 \rangle - \langle X \rangle^2$$

Poisson distribution (random distribution)

$$\sigma \approx \sqrt{\bar{X}}$$

Propagation of errors

$$\delta X = [\delta X_1^2 + \delta X_2^2 + \ldots]^{1/2}$$

Random errors: can be seen during multiple measurements and thus accounted for.

Systematic errors: errors that are intrinsic to the instrumentation and cannot be revealed by repeated measurements.
Particle Physics (3.7)

Hadrons: composite particles made of quarks

\[ \text{Mesons: 2 quarks} \]
\[ \pi^+, \pi^0, K^+, K^0, \gamma \]

\[ \text{Baryons: 3 quarks} \]
\[ p = uudd \]
\[ n = udds \]
\[ \Lambda^0, \Sigma^+ \]

Baryon \# = +1 for baryons, -1 for anti-baryons, 0 for non-baryons

Fermions: \( \frac{1}{2} \) integer spin

\[ \text{Leptons: spin} \frac{1}{2} \]
\[ e^-, \mu^-, \tau^-, \bar{e}, \bar{\mu}, \bar{\tau} \]

There are 3 lepton \#s
\[ L_e, L_\mu, L_\tau = \frac{1}{2}, -1, 0 \]

\[ \text{Quarks: spin} \frac{1}{2} \]
\[ \frac{1}{3} U, \frac{2}{3} C, \frac{1}{3} T \]
\[ -\frac{1}{3} D, -\frac{1}{3} S, -\frac{1}{3} B \]

Bosons: integer spin particles
\[ \gamma, g, \text{graviton, } W^+, W^-, Z^0 \]

Conservation Laws:

Energy & momentum
Parity = not in weak
Charge
Time symmetry
Strangeness = not in weak

Baryon \#:
Lepton \#
The Primitive Cell

\[ V_{\text{primitive cell}} = \frac{V_{\text{Bravais lattice}}}{\# \text{ of lattice points}} \]

simple cubic \( \rightarrow V_p = a^3 \)
1 lattice point inside

B.C.C. \( \rightarrow V_p = a^3/2 \)
8 lattice points

F.C.C. \( \rightarrow V_p = a^3/4 \)
4 lattice points

Resistivity of a semiconductor

\[ \rho \approx \frac{1}{r} \]

Effective Mass

\[ m^* = \frac{\hbar^2}{\frac{\partial^2 E}{\partial k^2}} \]

"Really" derivation:

\[ \rho = \frac{\hbar}{mk} \]
\[ E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \]
\[ \frac{\partial E}{\partial k^2} = \frac{\hbar^2}{m} \]
\[ \frac{1}{m} = \frac{\partial E}{\partial k^2} \frac{1}{\hbar^2} \]
\[ m = \frac{\hbar^2}{\frac{\partial^2 E}{\partial k^2}} \]
\[ \begin{align*}
\text{Math} \quad (\sim 3 \text{p}) \\
\text{gradient} \quad \nabla f &= \left( \frac{\partial f}{\partial x}, \ldots, \frac{\partial f}{\partial x_n} \right) \\
\text{divergence} \quad \nabla \cdot \vec{F} &= \frac{\partial F_i}{\partial x_i} + \ldots + \frac{\partial F_n}{\partial x_n} \\
\text{curl} \quad (\text{in } \mathbb{V}^3) \quad \nabla \times \vec{F} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\
\text{Divergence Theorem} \quad \iint_S \vec{F} \cdot dS &= \iiint_V \nabla \cdot \vec{F} \, dV \\
\text{Stoke's Theorem} \quad \text{(basic form for physics)} \quad \iint_{\partial A} \vec{F} \cdot d\vec{A} &= \iint_A \nabla \times \vec{F} \, d\vec{A} \\
\text{Important Identities} \quad \nabla \cdot (\nabla \times \vec{H}) &= 0 \\
\nabla \times (\nabla f) &= 0 \\
\text{BAC - CAB rule} \quad A \times (B \times C) &= B(A \cdot C) - C(A \cdot B) 
\end{align*} \]
Basic Equations from Astronomy / Astrophysics (~ 1/1)

$M_1 - m_2 = -2.5 \log_{10} \left( \frac{F_1}{F_2} \right)$  
Apparent magnitude

$M = m + 5 - 5 \log_{10} \left( \frac{d}{pc} \right) - A$  
Absolute magnitude

$\text{f/\text{v}} \propto \frac{1}{f}$

Magnification = $f_0 / f_{\text{eyepiece}}$

$\alpha_c = \frac{1.22 \lambda}{D}$  
Angular resolution / Rayleigh criterion

Image scale = $\frac{1}{f_{\text{tan} 1''}} = \frac{206,265}{f} \Rightarrow $ arcseconds/mm
Basic tips

→ If a problem looks hard/tedious, try process of elimination first. Use limiting cases to eliminate answers.

→ Very few problems are simply "plug into this equation" although a few are. Most problems combine at least 2 concepts or have a twist on them.

→ Dimensional analysis is occasionally useful.

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